Semester One Examination, 2021

Question/Answer booklet

MATHEMATICS SPECIALIST UNIT 1

SOLUTIONS

Section Two: Calculator-assumed

WA student number: In figures

In words

Your name

Time allowed for this section

Reading time before commencing work: ten minutes
Working time: one hundred minutes

Number of additional answer booklets used (if applicable):

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators, which can include scientific, graphic and Computer Algebra System (CAS) calculators, are permitted in this ATAR course examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	50	35
Section Two: Calculator-assumed	13	13	100	92	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answers to the specific question asked and to follow any instructions that are specific to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

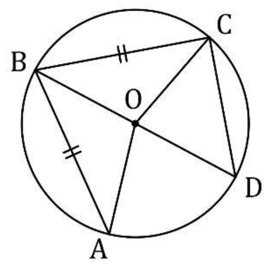
Section Two: Calculator-assumed **65% (92 Marks)**

This section has **thirteen** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

Question 9 **(5 marks)**

Points A, B, C and D lie on the circle with centre O as shown in the diagram, where $\angle A = 40°$, $AB = BC$ and BD is a diameter.



(a) Determine the size of $\angle AOD$. (2 marks)

Solution
Isosceles triangle: $\angle ABO = \angle A = 40°$ Angle on same arc: $\angle AOD = 2 \times 40° = 80°$
Specific behaviours
✓ indicates correct reasoning ✓ calculates angle

(b) Prove that $\Delta OAD \equiv \Delta ODC$. (3 marks)

Solution
Angle in semicircle: $\angle BAD = \angle BCD = 90°$ Hence $\Delta BAD \equiv \Delta BCD$ (RHS) and so $CD = AD$ (corresponding sides) Hence $\Delta OAD \equiv \Delta ODC$ (SSS) *(No need to show congruency of radii, diameter, etc)*
Specific behaviours
✓ establishes a pair of congruent triangles ✓ establishes congruent sides or angles ✓ states appropriate reason for congruency

Question 10 **(5 marks)**

Determine $\mathbf{p}$, the vector projection of

(a) a force of 320 N on a bearing $028°$ onto a force of 300 N on a bearing of $345°$. (3 marks)

Solution
$\lvert\mathbf{p}\rvert = 320\cos 43° = 234\text{ N}$ Hence $\mathbf{p}$ is a force of 234 N on a bearing of $345°$.
Specific behaviours
✓ calculates angle between vectors ✓ calculates magnitude ✓ states direction and magnitude

(b) $\mathbf{m}$ on $\mathbf{n}$ where $\mathbf{m} = (84, -13)$ and $\mathbf{n} = (14, -48)$. (2 marks)

Solution
$\mathbf{p} = \dfrac{\mathbf{m}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\mathbf{n}$ $= \dfrac{1800}{2500}\mathbf{n}$ $= (10.08, -34.56)$
Specific behaviours
✓ indicates method (possibly CAS) ✓ calculates vector

Question 11 **(8 marks)**

(a) State whether each of the following statements are true or false, supporting each answer with an example or counterexample.

(i) A quadrilateral with four congruent sides is a square. (2 marks)

Solution
False. Counterexample: rhombus.
Specific behaviours
✓ states false ✓ draws or names counterexample

(ii) The size of one interior angle of a regular polygon with at least five sides is always obtuse. (2 marks)

Solution
True. Interior angle of a regular hexagon is $120°$, an obtuse angle.
Specific behaviours
✓ states true ✓ example with obtuse angle calculated

(b) Consider the statement $\angle A \geq 90° \Rightarrow \angle B < 90°$ that refers to angles in triangle ABC.

(i) Write the converse of the statement in simplest form. (1 mark)

Solution
$\angle B < 90° \Rightarrow \angle A \geq 90°$
Specific behaviours
✓ correct converse

(ii) Write the contrapositive of the statement in simplest form. (1 mark)

Solution
$\angle B \geq 90° \Rightarrow \angle A < 90°$
Specific behaviours
✓ contrapositive that doesn't use 'not'

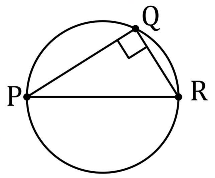
(iii) Briefly discuss the truth of the original statement, the converse statement, and the contrapositive statement. (2 marks)

Solution
The original statement is true and so is the contrapositive, by definition. However, the converse is false - when the triangle is acute, for example.
Specific behaviours
✓ states original and contrapositive true ✓ states converse false, with justification

Question 12 **(7 marks)**

(a) Points P, Q and R lie on a circle of radius 5 cm, so that PR is a diameter and $QR = 4$ cm. Determine the exact area of triangle PQR. (3 marks)

Solution



$$PR = 10, QR = 4 \Rightarrow BC = \sqrt{10^2 - 4^2} = \sqrt{84}$$

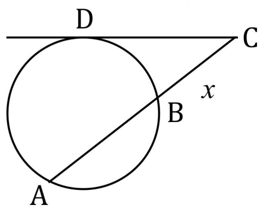
$$\text{Area} = \frac{1}{2}(4)2\sqrt{21} = 4\sqrt{21}\ \text{cm}^2$$

Specific behaviours

✓ indicates PQR is right triangle
✓ calculates missing side
✓ calculates area

(b) A secant meets a circle at points A and B, where $AB = 4$ cm. A tangent to the same circle at point D intersects the secant at point C, where $CD = 14$ cm. Given that $BC < AC$, determine the exact distance AC and the exact distance BC. (4 marks)

Solution



Using secant-tangent theorem:

$$14^2 = x(x + 4)$$

$$BC = x = 10\sqrt{2} - 2\ \text{cm}$$

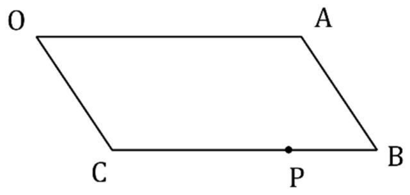
$$AC = BC + 4 = 10\sqrt{2} + 2\ \text{cm}$$

Specific behaviours

✓ sketch diagram
✓ formulates equation
✓ solves equation for positive distance
✓ calculates second distance

Question 13 **(6 marks)**

Parallelogram $OABC$ is shown where point P lies on side BC such that $BP:PC = 1:3$.



Point Q, not shown, lies on diagonal AC such that $AQ:QC = 4:1$.

Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$.

Express the following in terms of $\mathbf{a}$ and $\mathbf{c}$.

(a) $\overrightarrow{BO}$. (1 mark)

Solution
$\overrightarrow{BO} = -\mathbf{a} - \mathbf{c}$
Specific behaviours
✓ correct expression

(b) $\overrightarrow{AQ}$. (2 marks)

Solution
$\overrightarrow{AQ} = \frac{4}{5}\overrightarrow{AC} = \frac{4}{5}(\mathbf{c} - \mathbf{a})$
Specific behaviours
✓ uses ratio correctly ✓ correct expression

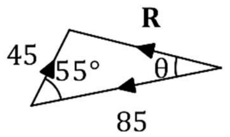
(c) $\overrightarrow{QP}$. (3 marks)

Solution
$\overrightarrow{QP} = \overrightarrow{QC} + \overrightarrow{CP}$ $= \frac{1}{5}(\mathbf{c} - \mathbf{a}) + \frac{3}{4}\mathbf{a}$ $= \frac{1}{5}\mathbf{c} + \left(\frac{3}{4} - \frac{1}{5}\right)\mathbf{a}$ $= \frac{1}{5}\mathbf{c} + \frac{11}{20}\mathbf{a}$
Specific behaviours
✓ expresses as sum of vectors ✓ expresses individual vectors correctly ✓ simplifies, using correct vector notation throughout

Question 14 **(8 marks)**

A small body is acted on by force $\mathbf{F_1}$ of 85 N on a bearing of $260°$ and by force $\mathbf{F_2}$ of 45 N on a bearing of $025°$.

(a) Sketch a diagram to show $\mathbf{F_1} + \mathbf{F_2}$ and their resultant $\mathbf{R}$. (2 marks)

Solution

Specific behaviours
✓ nose-to-tails vectors ✓ labels and angle

(b) Determine the magnitude and bearing of $\mathbf{R}$. (4 marks)

Solution
$r^2 = 45^2 + 85^2 - 2(45)(85)\cos 55° \Rightarrow r = 69.7$ N $\frac{\sin\theta}{45} = \frac{\sin 55°}{69.7} \Rightarrow \theta = 31.9°$ $260° + 32° = 292°$ Magnitude of $\mathbf{R}$ is 69.7 N and bearing is $292°$.
Specific behaviours
✓ expression using cosine rule with magnitude ✓ calculates magnitude ✓ expression using sine rule with angle ✓ calculates bearing

(c) Express $\mathbf{R}$ in component form $a\mathbf{i} + b\mathbf{j}$. (2 marks)

Solution
Angle of $\mathbf{R}$ from x-axis is $158.1°$. $\mathbf{R} = 69.7(\cos(158.1°)\,\mathbf{i} + \sin(158.1°)\,\mathbf{j})$ $= -64.7\mathbf{i} + 26.0\mathbf{j}$
Specific behaviours
✓ indicates method (possibly CAS) ✓ calculates components

Question 15 **(8 marks)**

Consider the set of integers between 2000 and 8000 inclusive.

(a) Show that there are 462 integers in this set that are a multiple of 13. (2 marks)

Solution
Number of multiples from 1 to upper bound: $n = \lfloor 8000 \div 13 \rfloor = 615$ Number of multiples from 1 to lower bound: $n = \lfloor 2000 \div 13 \rfloor = 153$ Hence $615 - 153 = 462$ multiples in interval.
Specific behaviours
✓ calculates multiples from 1 to lower, upper bounds ✓ calculates difference

(b) Determine the number of integers in this set that are

(i) a multiple of 13 and a multiple of 18. (3 marks)

Solution
$LCM(13, 18) = 234$ $n = \lfloor 8000 \div 234 \rfloor - \lfloor 2000 \div 234 \rfloor = 34 - 8 = 26$
Specific behaviours
✓ states LCM ✓ calculates multiples from 1 to lower, upper bounds ✓ calculates difference

(ii) not a multiple of 13 and not a multiple of 18. (3 marks)

Solution
Multiples of 18: $n = \lfloor 8000 \div 18 \rfloor - \lfloor 2000 \div 18 \rfloor = 444 - 111 = 333$ Multiples of 13 or 18: $n = 462 + 333 - 26 = 769$ Number of integers: $(8000 - 2000 + 1) - 769 = 6001 - 769 = 5232$
Specific behaviours
✓ multiples of 18 ✓ multiples of 13 or 18 ✓ correct number of integers

Question 16 **(8 marks)**

Each letter in the word CLOUDLESS is printed individually on a card. When cards are arranged next to each other in a line, determine the number of different permutations

(a) of all the cards. (2 marks)

Solution
Note: There are two L's and two S's. $$n = \frac{9!}{2!\,2!} = 90\,720$$
Specific behaviours
✓ expression that allows for repeated letters ✓ calculates number

(b) of all the cards where all the consonants are adjacent. (2 marks)

Solution
Note: There are six consonants that form a group to be arranged with the remaining three letters. $$n = \frac{4!\,6!}{2!\,2!} = 4320$$
Specific behaviours
✓ explains or clearly indicates grouping of consonants ✓ calculates number

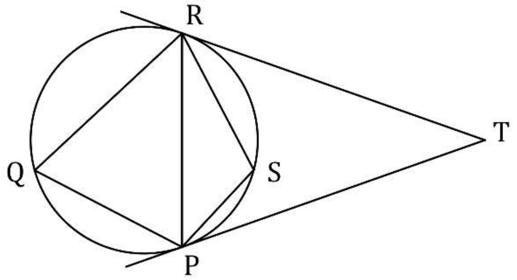
(c) using any 4 of the cards. (4 marks)

Solution
Count cases by selecting and then arranging: 1. All letters different $$n_1 = \binom{7}{4} \times 4! = 35 \times 24 = 840$$ 2. One pair (LL or SS) and two different: $$n_2 = 2 \times \binom{6}{2} \times \frac{4!}{2!} = 30 \times 12 = 360$$ 3. Two pairs (LL and SS): $$n_3 = 1 \times \frac{4!}{2!\,2!} = 1 \times 6 = 6$$ Number of permutations: $840 + 360 + 6 = 1206$
Specific behaviours
✓ identifies mutually exclusive cases ✓ counts one case correctly ✓ counts second case correctly ✓ counts all cases correctly and calculates total

Question 17 **(7 marks)**

(a) The diagram shows points P, Q, R and S on the circumference of a circle. Tangents to the circle from P and R meet at point T.

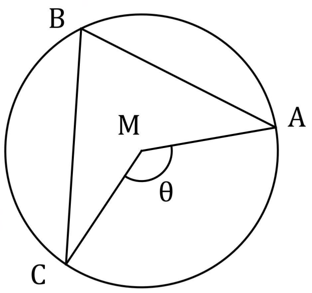
Given that $\angle T = 36°$, determine the size of $\angle Q$ and the size of $\angle S$.



(3 marks)

Solution
Isosceles triangle: $\angle PRT = \frac{1}{2}(180° - 36°) = 72°$
Alternate segment: $\angle Q = 72°$
Cyclic quadrilateral: $\angle S = 180° - 72° = 108°$
Specific behaviours
✓ shows use of at least two relevant circle theorems ✓ first angle ✓ second angle

(b) In the circle shown below $\angle A = 32°$, $\angle C = 24°$ and $\theta = 114°$. Prove by contradiction that M is not the centre of the circle. (4 marks)



Solution
Assume that M is the centre of the circle. Then $\angle ABM = \angle A = 32°$ (isosceles triangle) and $\angle CBM = \angle C = 24°$(isosceles triangle). Hence $\angle B = 32° + 24° = 56°$ and $\theta = 2 \times 56° = 112°$ (angles at centre - circumference). But this contradicts the initial information that $\theta = 112°$ and so the initial assumption must be wrong. Hence M is not the centre of the circle.
Specific behaviours
✓ clearly states assumption that M is centre of circle ✓ uses isosceles triangles to calculate $\angle B$ ✓ uses angle at centre - circumference theorem ✓ notes contradiction and draws conclusion

Question 18 **(8 marks)**

Small bodies P and Q are moving with constant velocities $(2, 0)$ m/s and $(1, -2)$ m/s respectively.

P has initial position vector $(1, -2)$ m and Q has initial position vector $(3, 4)$ m.

(a) Determine the distance between the bodies after five seconds. (3 marks)

Solution
Positions after five seconds: $$r_P = \begin{pmatrix}1\\-2\end{pmatrix} + 5\begin{pmatrix}2\\0\end{pmatrix} = \begin{pmatrix}11\\-2\end{pmatrix}$$ $$r_Q = \begin{pmatrix}3\\4\end{pmatrix} + 5\begin{pmatrix}1\\-2\end{pmatrix} = \begin{pmatrix}8\\-2\end{pmatrix}$$ $$\overrightarrow{PQ} = \begin{pmatrix}8\\-2\end{pmatrix} - \begin{pmatrix}11\\-2\end{pmatrix} = \begin{pmatrix}-3\\-4\end{pmatrix}$$ $$\left\lvert\overrightarrow{PQ}\right\rvert = 5 \text{ m}$$
Specific behaviours
✓ positions ✓ vector $\overrightarrow{PQ}$ ✓ distance

(b) Show that the distance between the bodies after t seconds is given by $\sqrt{5t^2 - 28t + 40}$. (3 marks)

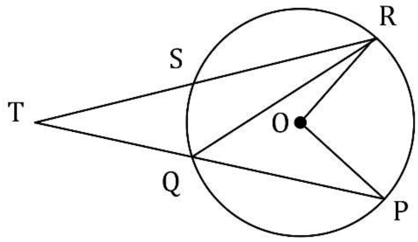
Solution
$$r_{PQ} = \begin{pmatrix}3\\4\end{pmatrix} + t\begin{pmatrix}1\\-2\end{pmatrix} - \begin{pmatrix}1\\-2\end{pmatrix} - t\begin{pmatrix}2\\0\end{pmatrix} = \begin{pmatrix}-t+2\\-2t+6\end{pmatrix}$$ $$\lvert r_{PQ}\rvert = \sqrt{(-t+2)^2 + (-2t+6)^2} = \sqrt{5t^2 - 28t + 40}$$
Specific behaviours
✓ vector $\overrightarrow{PQ}$ at time t ✓ simplifies vector ✓ expression for magnitude and simplifies

(c) Prove that the bodies do not meet. (2 marks)

Solution
Require $5t^2 - 28t + 40 = 0$: $$\Delta = b^2 - 4ac = (-28)^2 - 4(5)(40) = -16$$ Since the discriminant is negative, the distance can never be zero and hence the bodies never meet.
Specific behaviours
✓ states condition for bodies to meet ✓ justifies that condition never met

Question 19 **(7 marks)**

In the diagram shown, secants PQ and RS intersect at T, a point outside the circle with centre O.



(a) Determine the size of $\angle RQP$ and $\angle ROP$ when $\angle T = 36°$ and $\angle SRQ = 19°$. (2 marks)

Solution
$\angle RQP = 19° + 36° = 55°$ $\angle ROP = 2 \times 55° = 110°$
Specific behaviours
✓ first angle ✓ second angle

(b) Prove that when secants PQ and RS intersect at T, a point outside the circle with centre O, then $\angle T = \frac{1}{2}(\angle ROP - \angle SOQ)$. (4 marks)

Solution
Exterior angle of triangle: $\angle PQR = \angle T + \angle QRS$ Inscribed angles: $\angle PQR = \frac{1}{2}\angle ROP$ Inscribed angles: $\angle QRS = \frac{1}{2}\angle SOQ$ Substituting: $\angle T = \frac{1}{2}\angle ROP - \frac{1}{2}\angle SOQ$ Factoring: $\angle T = \frac{1}{2}(\angle ROP - \angle SOQ)$
Specific behaviours
✓ relation using exterior angles ✓ uses inscribed angles twice ✓ substitutes and factors ✓ notes reasoning throughout

(c) Determine the size of $\angle T$ when $\angle SOQ = 42°$ and $\angle ROP = 82°$. (1 mark)

Solution
$\angle T = \frac{1}{2}(82° - 42°) = 20°$
Specific behaviours
✓ correct angle

Question 20 **(7 marks)**

(a) A manufacturer makes the same plastic toy figure in nine different colours and sells them in packs of four. The toys inside each pack are randomly chosen from the production line in such a way that all are of a different colour.

Determine the least number of packs that a retailer should buy from the manufacturer to be certain of obtaining at least five packs containing the same colour combination of toys.

(3 marks)

Solution
There are $\binom{9}{4} = 126$ different packs. Using the pigeonhole principle with the number of different packs as pigeonholes (126) and the number bought by the retailer as pigeons (n): $\lceil n \div 126 \rceil = 5 \Rightarrow n = 126 \times 4 + 1 = 505$ The retailer must buy at least 505 packs.
Specific behaviours
✓ calculates different number of packs ✓ applies pigeonhole principle ✓ correct least number

(b) A set of cards is numbered with all the integers from 1 to 18 inclusive. The cards are shuffled, placed face down and then the cards turned over one by one.

Determine how many cards must be turned over to be certain that at least one of the numbers on a face up card will be exactly half of the number on another face up card.

(4 marks)

Solution
Partition integers (pigeons) into pigeonholes, with any pair meeting given condition in same pigeonhole (possible example): $\{1,2\}\ \{3,6\}\ \{4,8\}\ \{5,10\}\ \{7,14\}\ \{9,18\}\ \{11\}\ \{12\}\ \{13\}\ \{15\}\ \{16\}\ \{17\}$ There are 12 pigeonholes and so $12+1$ pigeons are required. 13 cards must be turned over to be certain.
Specific behaviours
✓ treats integers as pigeons ✓ identifies pigeonholes ✓ indicates use of pigeonhole principle ✓ correct number

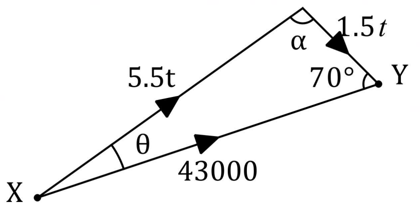
Question 21 **(8 marks)**

Harbour Y lies on a bearing of $065°$ from harbour X and the straight line distance between the harbours is 43 km. Between the harbours, a steady current is moving in a south easterly direction at a speed of 1.5 metres per second.

A boat with a cruising speed of 5.5 metres per second is to travel from harbour X to harbour Y in the least possible time.

(a) Sketch a diagram, roughly to scale, to show the resultant of the sum of the displacement vectors of the boat and the current. (2 marks)

Solution



Specific behaviours

✓ shows vector triangle, roughly to scale
✓ labels X and Y, sides, indicates angle

(b) Determine the bearing it should steer, to the nearest degree, and the time its journey takes, to the nearest minute. (6 marks)

Solution

$$\frac{\sin\theta}{1.5t} = \frac{\sin 70°}{5.5t} \Rightarrow \theta = 14.85°$$

$$\alpha = 180° - 70° - 14.85° = 95.15°$$

$$\frac{5.5t}{\sin 70°} = \frac{43\ 000}{\sin 95.15°} \Rightarrow t = 7376\ \text{s}$$

$$7376 \div 60 = 122.9\ \text{min}$$

$$065° - 14.85° \approx 050°$$

Boat should steer on bearing of $050°$ and will reach Y after 2 hours and 3 minutes.

Specific behaviours

✓ equation involving θ
✓ solves for θ
✓ equation involving t
✓ solves for t
✓ calculates and states bearing
✓ states time, to nearest minute

End of questions

Supplementary page

Question number: _________

Supplementary page

Question number: _________

Supplementary page

Question number: _________

Supplementary page

Question number: _________